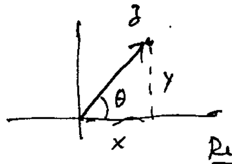
Basics of Complex numbers

The complex number $z = x + jy$ can be represented by the point (x, y) on the x-y plane, or the vector from $(0,0)$ to (x, y)



$|z| = \sqrt{x^2+y^2}$ is called its magnitude

$\theta = \tan^{-1}\frac{y}{x}$ is called its argument

Real part of z is x, imaginary part of z is y

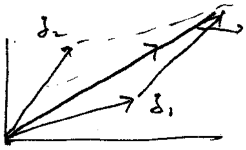
Operations with complex numbers follow the rule of ~~arithmetics~~ algebra with the proviso that $j^2 = -1$.

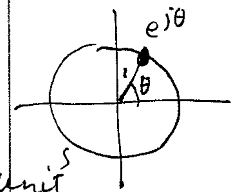
Examples ① $(2+3j)(1-j) = 2 - 2j + 3j + 3j(-j) = 2 + j + 3 = 5 + j$

② $\dfrac{2+3j}{1-2j} = \dfrac{(2+3j)(1+2j)}{(1-2j)(1+2j)} = \dfrac{2+4j+3j-6}{1+4} = -\frac{4}{5} + \frac{7}{5}j$

③ $\frac{1}{j} = -j$

Addition of ~~two~~ complex numbers can be interpreted as vector addition on the complex plane:





Unit circle

$e^{j\theta} = \cos\theta + j\sin\theta$

$$e^{j\theta_1} \cdot e^{j\theta_2} = (\cos\theta_1 + j\sin\theta_1)(\cos\theta_2 + j\sin\theta_2)$$
$$= (\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + j(\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2)$$
$$= \cos(\theta_1+\theta_2) + j\sin(\theta_1+\theta_2) = e^{j(\theta_1+\theta_2)}$$

$\therefore$ $e^{j\theta}$ behaves like exponential function

$e^{-j\theta} = \dfrac{1}{e^{j\theta}}$, $\quad \dfrac{d}{d\theta}e^{j\theta} = je^{j\theta}$. $\quad \dfrac{d}{dt}e^{j\omega t} = j\omega e^{j\omega t}$

A complex number $z = x + yj$ can be written in the polar form

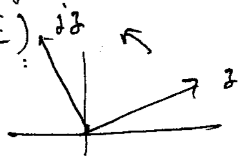
$z = Ae^{j\theta}$ $\quad A = \sqrt{x^2+y^2} = |z|$, $\quad \theta = \tan^{-1}\frac{y}{x}$.

Examples ① $2 - 3j = \sqrt{2^2+3^2}\,e^{-j\phi}$ $\quad \phi = \tan^{-1}\frac{3}{2} = 0.98$ rad.

$= 3.6\,e^{-0.98j}$

② $j = e^{j\frac{\pi}{2}}$ $\quad -j = e^{-j\frac{\pi}{2}}$.

Multiplication of a complex number by j ~~can~~ corresponds to rotation by $90°$: $\quad jz = e^{j\frac{\pi}{2}}Ae^{j\theta} = Ae^{j(\theta+\frac{\pi}{2})}$.



If $z_1 = A_1e^{j\theta_1}$, $z_2 = A_2e^{j\theta_2}$

$z_1z_2 = A_1A_2e^{j(\theta_1+\theta_2)}$ $\quad \dfrac{z_1}{z_2} = \dfrac{A_1}{A_2}e^{j(\theta_1-\theta_2)}$

Use of complex numbers in AC Circuits

Any sinusoidally oscillating quantity

$$A = A_{max}\cos(\omega t+\phi) = \text{real part of } A_{max}e^{j(\omega t+\phi)}$$

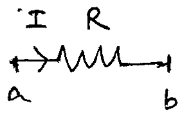
$$= Re[A_{max}e^{j\phi}\cdot e^{j\omega t}] = Re(a e^{j\omega t})$$

where $a = A_{max}e^{j\phi}$ will be used to represent A.

To recover A from a, just take the real part of $a e^{j\omega t}$

<u>Complex voltage-current relations</u>

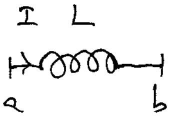
$$I = I_{max}\cos(\omega t+\phi) \longrightarrow i = I_{max}e^{j\phi}$$



① Resistor

$$V = V_a - V_b = RI = RI_{max}\cos(\omega t+\phi) = Re[RI_{max}e^{j\phi}\cdot e^{j\omega t}]$$

$$= Re[\underbrace{Ri}_{v}\, e^{j\omega t}] \qquad \therefore v = Ri$$



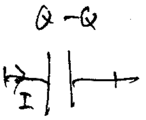
② Inductance

$$V = L\frac{dI}{dt} = -\omega L I_{max}\sin(\omega t+\phi) = \omega L I_{max}\cos(\omega t+\phi+\tfrac{\pi}{2})$$

$$= Re[\omega L I_{max}e^{j(\phi+\frac{\pi}{2})}e^{j\omega t}] = \cancel{Re[\omega L I_m}$$

$$= Re[\omega L e^{j\frac{\pi}{2}}\cdot I_{max}e^{j\phi}\cdot e^{j\omega t}]$$

$$= Re[\underbrace{j\omega L\cdot i}_{v}\cdot e^{j\omega t}] \qquad \therefore v = j\omega L i$$



③ Capacitor

$$V = \frac{Q}{C} = \frac{1}{C}\int I dt = \frac{1}{\omega C}I_{max}\sin(\omega t+\phi)$$

$$= \frac{1}{\omega C}I_{max}\cos(\omega t+\phi-\tfrac{\pi}{2}) = Re[\frac{1}{\omega C}I_{max}e^{j(\phi-\frac{\pi}{2})}e^{j\omega t}]$$

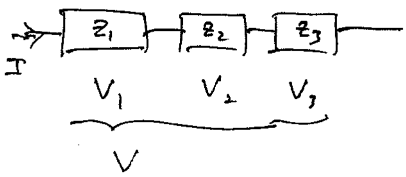
$$= Re[\frac{1}{j\omega C}\cdot i\cdot e^{j\omega t}] \qquad v = \frac{1}{j\omega C}i$$

Thus, in general, $v = z i$ where

	resistor	inductor	capacitor
z	R	$j\omega L$	$\frac{1}{j\omega C}$

Instead of using lower case v, z, i, we can use V, Z, I with the understanding that real parts are to be taken at the end.

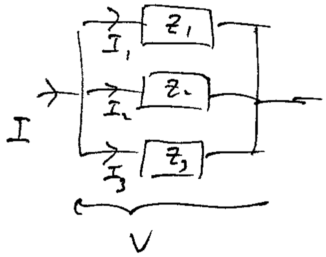
Impedance in series



$$V_1 = I Z_1$$
$$V_2 = I Z_2$$
$$V_3 = I Z_3$$

$$\therefore V = V_1 + V_2 + V_3 = I(Z_1 + Z_2 + Z_3) = I Z$$
$$Z = Z_1 + Z_2 + Z_3$$

Impedance in parallel

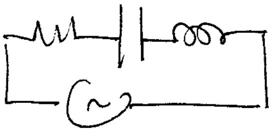


$$I_1 = \frac{V}{Z_1} \qquad I_2 = \frac{V}{Z_2} \qquad I_3 = \frac{V}{Z_3}$$

$$I = I_1 + I_2 + I_3 = V\left(\frac{1}{Z_1} + \frac{1}{Z_2} + \frac{1}{Z_3}\right) = \frac{V}{Z}$$

$$\frac{1}{Z} = \frac{1}{Z_1} + \frac{1}{Z_2} + \frac{1}{Z_3}$$

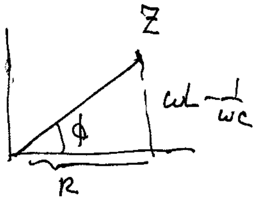
<u>Example 1</u> RLC circuit



$$\mathcal{E} = \mathcal{E}_{max} \cos\omega t = \mathrm{Re}(\underbrace{\mathcal{E}_{max}}_{\mathcal{E}} e^{j\omega t})$$

$$Z = R + j\omega L + \frac{1}{j\omega C} = R + j\left(\omega L - \frac{1}{\omega C}\right)$$

$$I = \frac{\mathcal{E}}{Z}$$



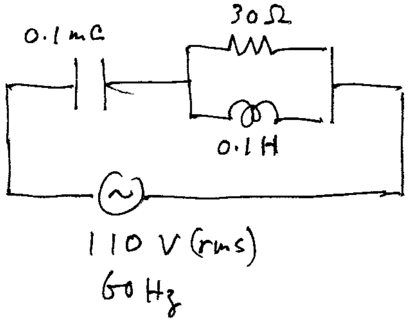
In polar form $Z = \sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}\, e^{j\phi}$

$$\tan\phi = \frac{\omega L - \frac{1}{\omega C}}{R}$$

$$\therefore I = \frac{\mathcal{E}}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}} e^{-j\phi}$$

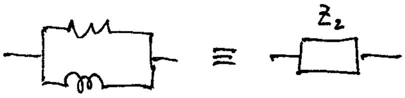
Actual current $= \dfrac{\mathcal{E}_{max}}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}} \cos(\omega t - \phi)$

Example 2. Find I_{rms} and power delivered by generator in the circuit



$\omega = 2\pi \times 60 = 377$ rad/s

Equivalent impedance of 30Ω & 0.1 H in parallel is given by

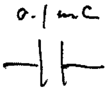


$$\frac{1}{Z_2} = \frac{1}{R} + \frac{1}{j\omega L} = \frac{j\omega L + R}{j\omega L R} = \frac{jX_L + R}{jX_L R}$$

$$\therefore Z_2 = \frac{jX_L R}{R + jX_L} \qquad X_L = \omega L = 377 \times 0.1 = 37.7\,\Omega$$

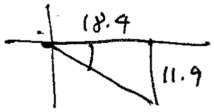
$$= \frac{jX_L R(R - jX_L)}{(R + jX_L)(R - jX_L)} = \frac{jX_L R^2 + X_L^2 R}{R^2 + X_L^2} = \frac{RX_L}{R^2 + X_L^2}(X_L + jR)$$

$$= 18.4 + 14.6j$$

for 

, $Z_1 = \frac{1}{j\omega C} = -\frac{j}{\omega C} = -26.5j$

$\therefore$ Equivalent impedance of circuit is



$$Z = Z_1 + Z_2 = 18.4 - 11.9j = |Z| e^{-j\delta}$$

$$|Z| = \sqrt{18.4^2 + 11.9^2} = 21.9\,\Omega \qquad \delta = \tan^{-1}\frac{11.9}{18.4} = 32.9^\circ = 0.584 \text{ rad}$$

$$I = \frac{\mathcal{E}}{Z} = \frac{\mathcal{E}}{|Z|e^{-j\delta}} = \frac{\mathcal{E}}{|Z|}e^{j\delta} = \frac{110}{21.9}e^{j\delta} = 5.02\, e^{j\delta}$$

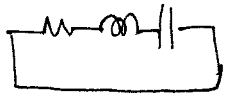
$$\therefore \quad I_{rms} = 5.02 \text{ A} \qquad I_{max} = \sqrt{2}\, I_{rms} = 7.10 \text{ A}$$

Real current is $I = 7.10 \cos(\omega t + \delta) = I_{max}\cos(\omega t + \delta)$

Power delivered by generator is

$$\langle \mathcal{E} I \rangle = \mathcal{E}_{rms} I_{rms} \cos\delta = 110 \times 5.02 \times \cos 32.9^\circ = 464 \text{ W}$$

Example 3 RLC circuit without generator



from $\mathcal{E} = IZ$ follows $Z = 0$ if $\mathcal{E} = 0$

$R + j\omega L + \frac{1}{j\omega C} = 0$ is an equation for ω.

$$j\omega CR - \omega^2 LC + 1 = 0$$

$$\omega^2 - \frac{jR}{L}\omega - \frac{1}{LC} = 0$$

$$\omega = +\frac{jR}{2L} \pm \sqrt{\left(-\frac{jR}{2L}\right)^2 + \frac{1}{LC}} = \pm\sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}} + j\frac{R}{2L}$$

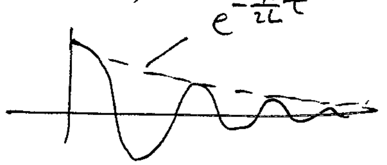
Case 1 $\frac{1}{LC} > \frac{R^2}{4L^2}$, (underdamped)

$\omega_0 = \sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}}$ is a real number

$$I = I_{max} e^{j\omega t} = I_{max} e^{j(\pm\omega_0 + j\frac{R}{2L})t} = I_{max} e^{-\frac{R}{2L}t} \cdot e^{\pm j\omega_0 t}$$

Real part of I is $I_{max} e^{-\frac{R}{2L}t} \cos(\omega_0 t)$

which is a damped oscillation



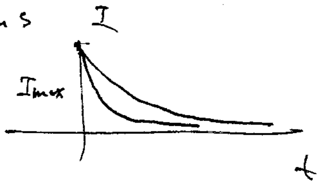
Case 2 $\frac{1}{LC} < \frac{R^2}{4L^2}$ (over damped)

$$\sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}} = \sqrt{\left(\frac{R^2}{4L^2} - \frac{1}{LC}\right)(-1)} = \sqrt{\frac{R^2}{4L^2} - \frac{1}{LC}} \cdot j \qquad j = \sqrt{-1}$$

$\therefore\ \omega = j\left[\frac{R}{2L} \pm \sqrt{\frac{R^2}{4L^2} - \frac{1}{LC}}\right] = j\gamma_1,\ j\gamma_2$ two solutions $\gamma_1, \gamma_2 > 0$

$\therefore\ I = I_{max} e^{j\omega t} = I_{max} e^{-\gamma_1 t}$ or $I_{max} e^{-\gamma_2 t}$

or any combinations



Case 3 $\frac{1}{LC} = \frac{R^2}{4L^2}$ (critically damped)

$$\omega = j\frac{R}{2L}$$

$$I = I_{max} e^{-\frac{R}{2L}t}$$