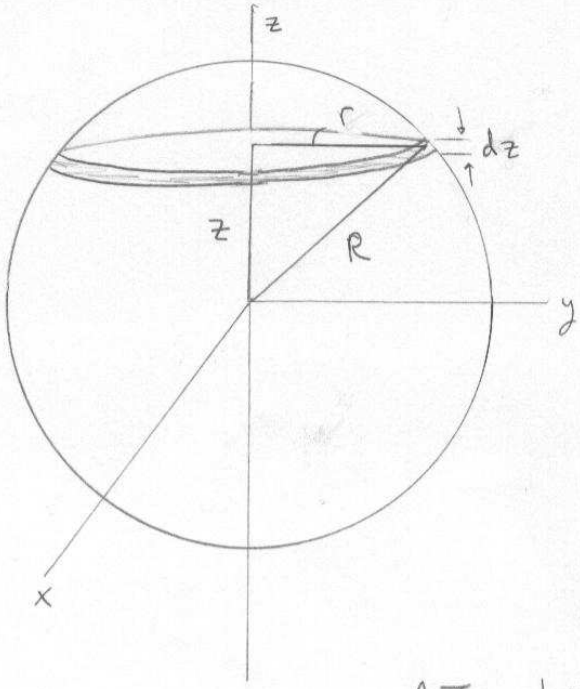
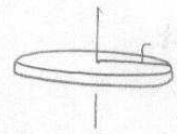
MOMENT of Inertia of a SOLID SPHERE



Using symmetry



Disk

$I = \frac{1}{2} M r^2$

$dI = \frac{1}{2} r^2 dm$

$\rho = \frac{m}{V} \qquad m = \rho V$

$dm = \rho\, dV$

and $dV = \pi r^2 dz$

$$dI = \frac{1}{2} r^2 dm = \frac{1}{2} r^2 (\rho\, dV) = \frac{1}{2} r^2 (\rho \pi r^2 dz)$$

$$dI = \rho \frac{1}{2} \pi r^4 dz \qquad \text{but} \qquad z^2 + r^2 = R^2$$

so $r^2 = R^2 - z^2$

$$dI = \rho \frac{1}{2} \pi (R^2 - z^2)^2 dz$$

$$I = \int dI = \rho \int_{z=-R}^{z=R} \frac{1}{2} \pi (R^2 - z^2)^2 dz = 2\rho \left[\frac{1}{2} \pi \int_0^R (R^2 - z^2)^2 dz \right]$$

$$I = \rho \pi \int_0^R (R^4 - 2R^2 z^2 + z^4)\, dz = \rho \pi \left[R^4 z - 2R^2 \frac{z^3}{3} + \frac{z^5}{5} \Big|_0^R \right]$$

$$I = \pi \rho \left(R^5 - 2R^2 \frac{R^3}{3} + \frac{R^5}{5} \right) = \rho \pi R^5 \left(1 - \frac{2}{3} + \frac{1}{5} \right) = \pi \rho R^5 \frac{8}{15}$$

where $\rho = \dfrac{M}{\frac{4}{3}\pi R^3}$ so $I = \dfrac{\pi R^5 8}{15} \left(\dfrac{M}{\frac{4}{3}\pi R^3} \right) = \boxed{\frac{2}{5} M R^2}$ QED