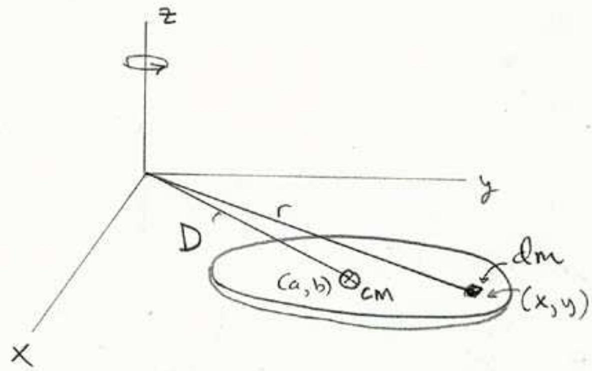
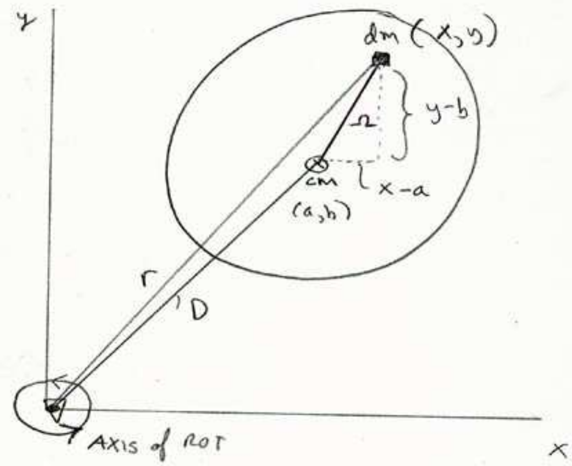
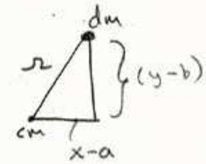
Proof of ∥-axis Th$^{\underline{m}}$ $\quad I_{PA} = I_{cm} + MD^2$





dm located at : (x,y)

CM " " : (a,b)



$$I_{PA} = \int r^2 dm \quad \text{where } r^2 = x^2 + y^2$$

but $\; \rho^2 = (x-a)^2 + (y-b)^2$

$$\rho^2 = x^2 - 2ax + a^2 + y^2 - 2by + b^2 \quad , \quad x^2 + y^2 = \rho^2 + 2ax + 2by - (a^2+b^2)$$

$$\text{so } I_{PA} = \int r^2 dm = \int (x^2+y^2)\,dm = \int \left[\left(\rho^2 + 2ax + 2by - (a^2+b^2) \right) \right] dm$$

$$I_{PA} = \underset{①}{\int \rho^2 dm} + \underset{②}{2a \int x\,dm} + \underset{③}{2b \int y\,dm} - \underset{④}{(a^2+b^2)\int dm}$$

#1 $\int \rho^2 dm = I_{cm}$ by definition of I_{cm}.

#2 note: $X_{cm} = \dfrac{\int x\,dm}{M}$, $\int x\,dm = M X_{cm}$, so $2a \int x\,dm = 2a \left[M X_{cm} \right] = 2Ma^2$ since $X_{cm} = a$

#3 $y_{cm} = \dfrac{\int y\,dm}{M}$, $\int y\,dm = M y_{cm}$, so $2b \int y\,dm = 2b \left[M y_{cm} \right] = 2Mb^2$, since $y_{cm} = b$

④ $\int dm = M$

so $I_{PA} = 1 + 2 + 3 + 4$

$$I_{PA} = I_{cm} + 2Ma^2 + 2Mb^2 - (a^2+b^2)M$$

$$= I_{cm} + Ma^2 + Mb^2$$

$$= I_{cm} + M(a^2+b^2)$$, but $a^2+b^2 = D^2$

so $$I_{PA} = I_{cm} + MD^2$$ QED.