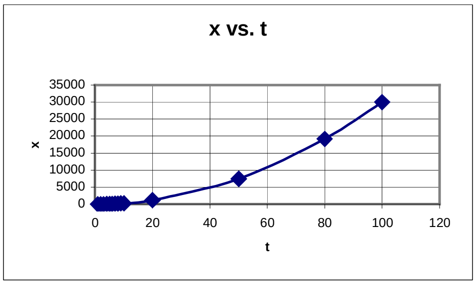
Graphing on log-log Paper

Suppose you were presented with the set of data shown below. A graph of x vs. t is also shown, and you can see it's a smooth curve. But other than that, it's not very informative. Suppose, however, in addition, there were reasons to believe that this data obeyed a power-law, $x = kt^n$, meaning that the function could be of the form $x = 8t^3$, where k = 8 and n = 3. How could you determine if this were true and, if it were, find the constants k and n?

Perhaps this function is $x = t^2$, or $x = 5t^4$. Actually, it is probably impossible to determine the exponent '*n*' and constant '*k*' by looking at this graph.



t (s)	x (m)
1	3
2	12
3	27
4	48
5	75
6	108
7	147
8	192
9	243
10	300
20	1200
50	7500
80	19200
100	30000

Your data

Data table (1)

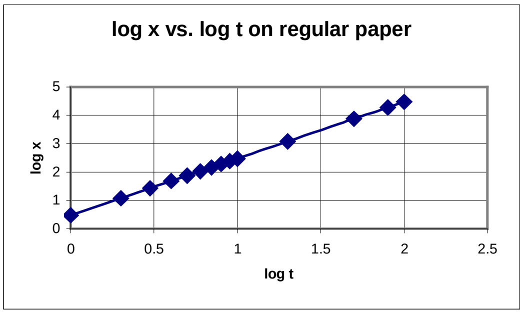
Here is a slick technique to solve this dilemma. Let's take the 'log' of both sides of our function:

Eq. (1) **log**(x) = **log** (k t^n) Recall: log AB = log A + log B and log A^n = n log A

So equation (1) becomes: $\log x = n \log t + \log k$ eq. (2)

But this has the form: y = m x + b , a straight line!

This means that we can just take the log of each data point and plot it on regular graph paper:



*** Note that the axes are **log x** and **log t** !!***

log of the data

log t	log x
0	0.477121
0.30103	1.079181
0.477121	1.431364
0.60206	1.681241
0.69897	1.875061
0.778151	2.033424
0.845098	2.167317
0.90309	2.283301
0.954243	2.385606
1	2.477121
1.30103	3.079181
1.69897	3.875061
1.90309	4.283301
2	4.477121

Data table (2)

How to find 'n' : Now we have a straight line whose slope is the exponent '*n*' and whose x-intercept is 'log *k*'. You can use this data table to show that the slope is '2'. Thus *n* = 2. Notice that since logs have no units, then the slope has no units.

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How to find 'k': The constant '*k*' is a little bit trickier. Just as you would find the y-intercept in y = mx + b by setting x = 0, you would find *k* by setting *n* log *t* equal to zero in equation (2).

Eq. (2) $\log x = n \log t + \log k$ So equation (2) becomes

$$\log x = \underline{n \log t} + \log k, \quad \text{or}$$

$$\log x = 0 + \log k, \qquad \text{thus} \quad \log x = \log k.$$

Now look on the second graph to see where the line crosses the *log x* axis. This occurs when log *x* = 0.477121. (* Remember, that value is ***not*** *x*, it's **log *x*** .)

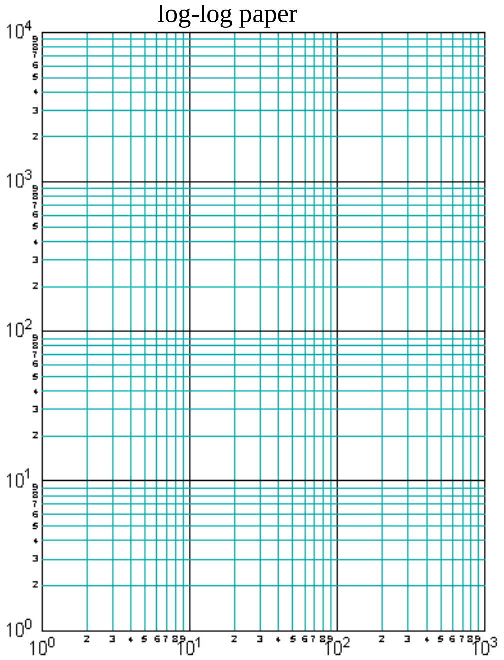
So log *x* = log *k*, and we have 0.477121 = log k . Solving for *k* yields *k* = 3.

So you have now found the constants *k* and *n* for the function $x = kt^n$. You can now state that the data of x vs. t can be described by the function $x = 3t^2$.

The easy way…

Taking the log of all the data and re-plotting it is tedious and time consuming. Fortunately there is an easier way! Instead of using your calculator to take the log of each data point, we can use special graph paper called logarithmic graph paper. Since the log of both variables *x* and *t* are needed, we can use log-log paper – it is just graph paper in which both axes are ruled logarithmically.

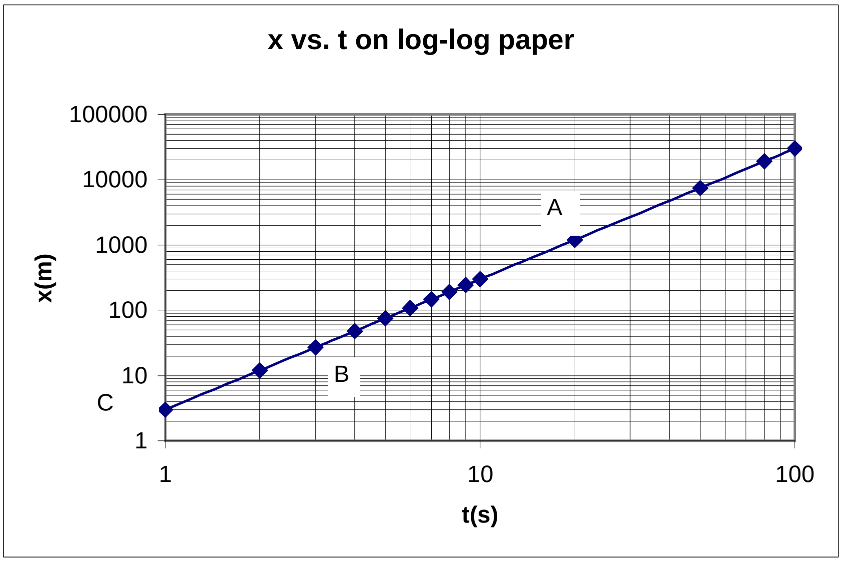
log-log paper



Powers of 10 graph paper
(log-log paper)

Notice that the log axes runs in exponential cycles. Each cycle runs linearly in 10's but the increase from one cycle to another is an increase by a factor of 10. So within a cycle you would have a series of: **10**, 20, 30, 40, 50, 60, 70, 80, 90, **100** (this could also be **1-10**, or **0.1-1**, etc.). The next cycle actually begins with **100** and progresses as 200, 300, 400, 500, 600, 700, 800, 900, **1000**. The cycle after that would be **1000**, 2000, 3000, 4000, 5000, 6000, 7000, 8000, 9000, **10000** and so on. So you see, the graph paper actually takes the *log* for you!

Now we will plot our original data from table (1) on log-log paper to see if has the form :
$\log x \;=\; n \log t \;+\; \log k.$



Since this produces a straight line, we know that the data must describe a function of the form $x = kt^n$ with slope *n* and vertical intercept *log k*.
Care must be taken when calculating the slope. Any number taken from the graph comes off the graph paper as the log of that number.

Calculation of the slope *n* :

Care must be taken when calculating the slope on log paper. Remember that you did not take the log of your data using your calculator, rather the logarithmic graph paper took the log for you! Thus, any number taken off the graph comes off the graph paper as the log of that number.

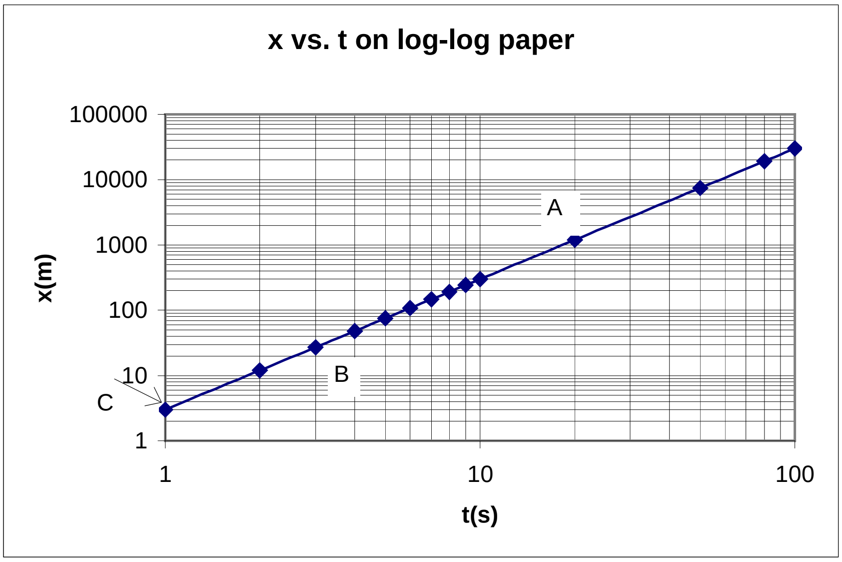
Method: Pick two points ***on your line*** : (x_{c2}, t_{c2}) and (x_{c1}, t_{c1})**.** Remember that your data points may not all be on the line. Even so, *you* must pick two points **on your line**!

$$\textit{the slope}: \quad n = \frac{\log(x_{c2}) - \log(x_{c1})}{\log(t_{c2}) - \log(t_{c1})} = \frac{\log\left(\frac{x_{c2}}{x_{c1}}\right)}{\log\left(\frac{t_{c2}}{t_{c1}}\right)}$$

For this example, I will use points A and B to calculate the slope. (* Remember that your data points may not all be on the line. Even so, *you* must pick two points **on your line**!)

$$\text{Slope } n = \frac{\log(1200) - \log(27)}{\log(20) - \log(3)} = \frac{\log(1200/27)}{\log(20/3)} = 2$$

Calculation of the constant *k:*



Just do the same as you would when solving for *b* in y = mx + b, namely, set x = 0 to get y = b.

We have log x = nlog t + log k.

So setting n log t = 0 leaves log x = log k.

Recall log t = 0 when t = 1 !

So look to see where your line intercepts the x-axis when *t* = 1 (not when t = 0, since log 0 is undefined there!)

In the graph above, the line intercepts the x-axis at point C. Point C corresponds to the value '3'.

Thus the x-intercept is log 3. Remember that it is ***not*** just 3 since the vertical axis is logarithmic,

so log *x* = log *k* becomes

log 3 = log *k*, and thus *k* = 3.

Therefore, we can write that this data fits the equation $x = 3t^2$.

Units of *k* : You must include the proper units with the value of *k*. To find them, simply rearrange the equation $x = kt^n$ to solve for *k*, in other words,

$k = x/t^n$.

Since *x* has units of meters and *t* has units of seconds, the units for *k* must be m/s^n. In this exercise where *n* = 2, *k* has units of m/s^2. So to fully express the function, we have:

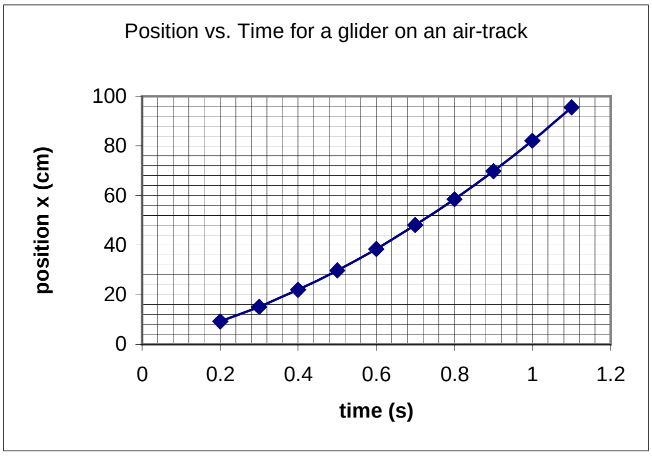
$$x = 3\frac{m}{s^2}t^2$$

Step-by-Step Tutorial to Assist You When Graphing on Log Paper

I hope you agree that the method of graphing your data on log-log paper greatly simplifies the process of determining the functional dependence between two variables.
The techniques of graphing on logarithmic paper are valuable tools that you will need in your career as a scientist or engineer. You will be expected to make these types of graphs several times throughout this semester. If you ask me “How do I graph on log paper?”, I will refer you to this worksheet tutorial.

More Examples:
Here is a student’s data for a trial run of a glider moving down a tilted airtrack.

Time t (s)	Position x (cm)
0.1	4.1
0.2	9.1
0.3	15.1
0.4	21.9
0.5	29.7
0.6	38.35
0.7	48
0.8	58.45
0.9	69.7
1	82.1
1.1	95.5

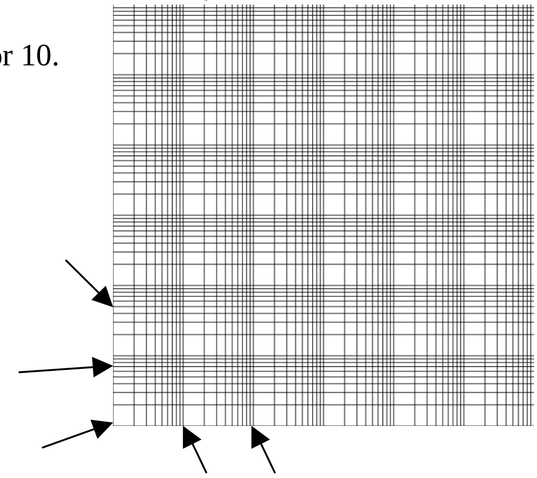


It appears that the data obeys a power law of the form $x = k\,t^n$, where k is a constant to be determined. We will now use the techniques outlined in the lab session to determine the constants k and n in a step-by-step manner.

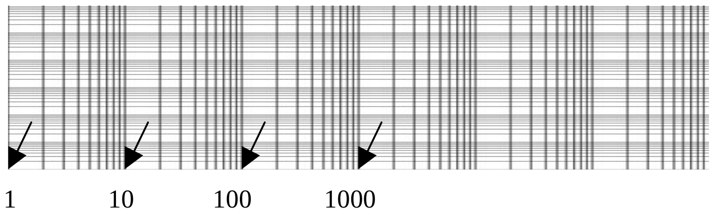
Step 1: Choosing a scale for the *x* and *t* axes:

Remember that the log paper is ruled in powers (or decades) of 10 as indicated by the arrows.

This means that the axes must be labeled in successive powers or 10.

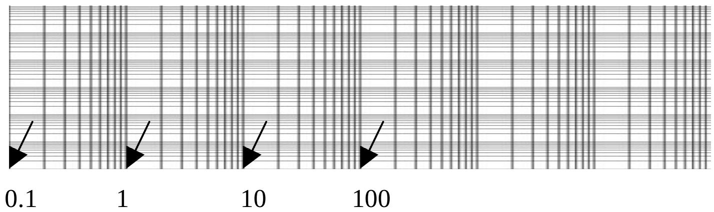


For example: the horizontal axis could be labeled as



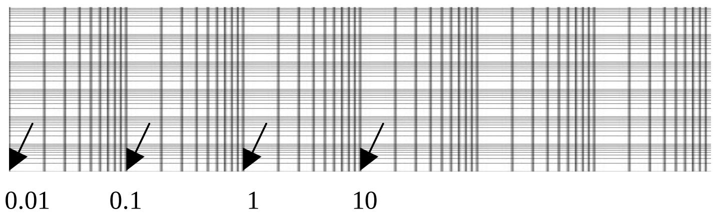
Where we used: 10^0, 10^1, 10^2, 10^3

or,



Where we used: 10^{-1}, 10^0, 10^1, 10^2

or,

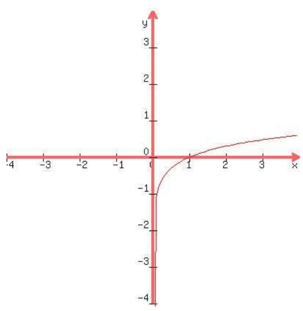


Where we used: 10^{-2}, 10^{-1}, 10^0, 10^1.
All of which are successive powers of 10.

Answer this question: Where is the zero located on the log paper?
Your answer: ____

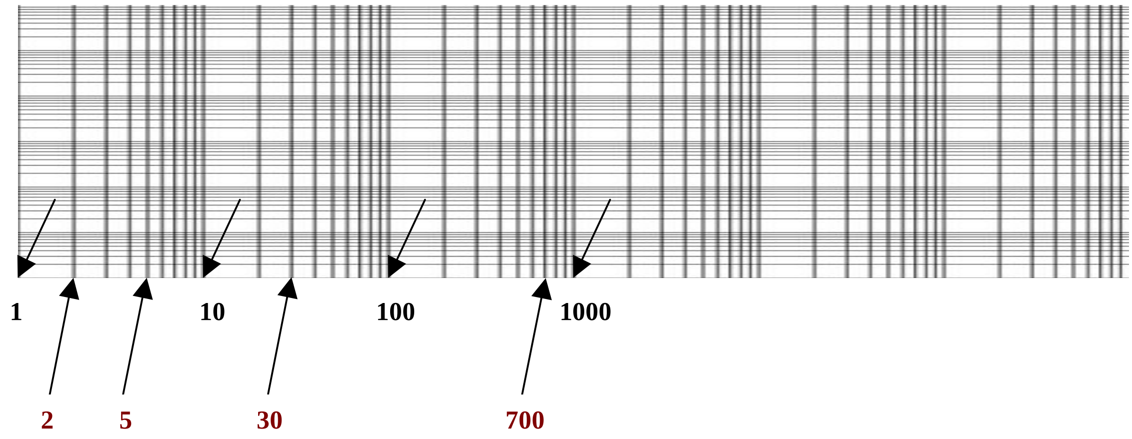
Notice that the lines in between the powers of 10 *are not evenly spaced*! Why is this??!!
It's because the scale on the log paper represents a function, the logarithmic function.

Recall $y = \log x$ looks like

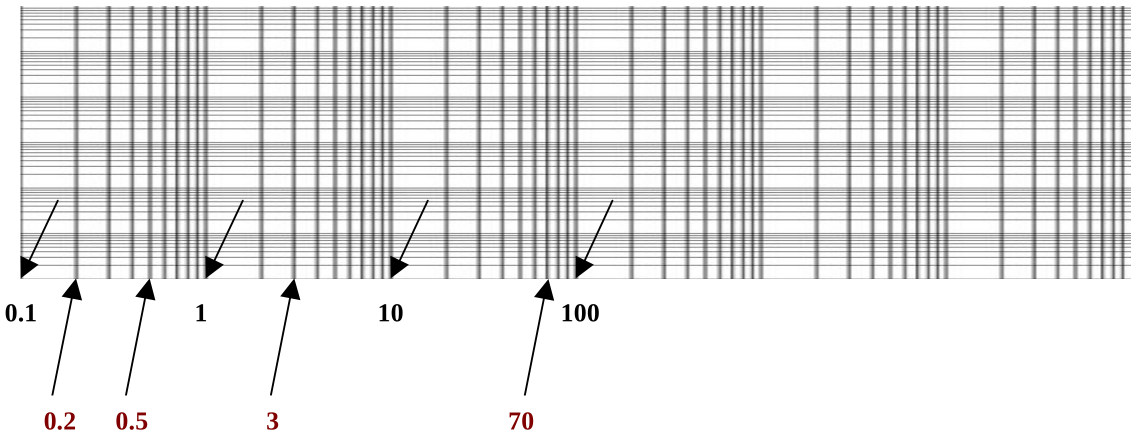


The function is NOT LINEAR, so that is why the in-between lines on the log paper are not evenly spaced.

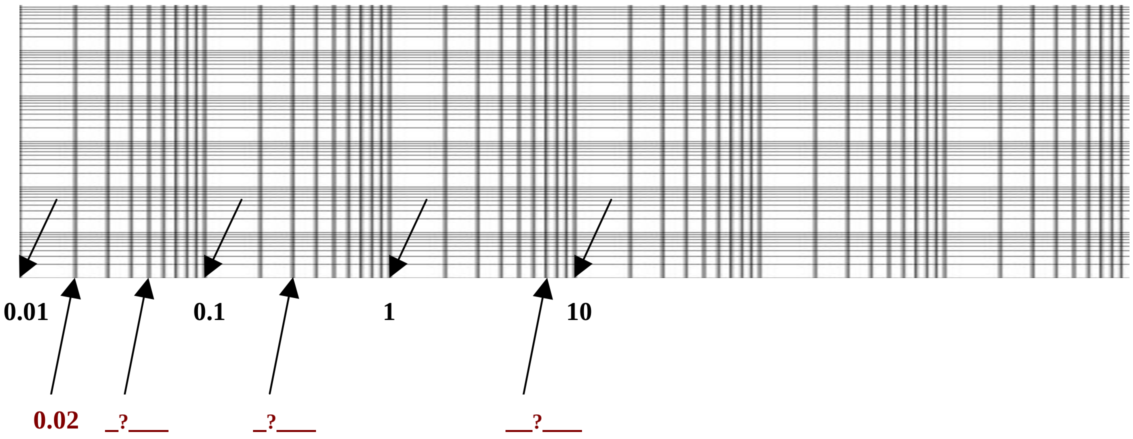
Let's look at how you would read some of these lines. For example:



Or,



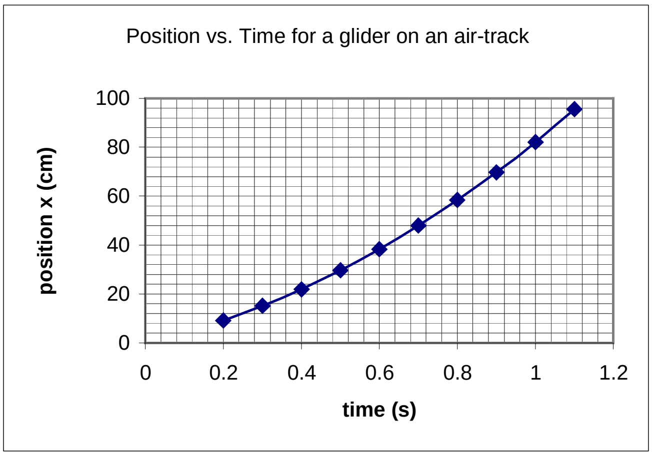
Try this for yourself:



You should have filled in: 0.05, 0.3, and 7.

Let's go back to our original example:

Time t (s)	Position x (cm)
0.1	4.1
0.2	9.1
0.3	15.1
0.4	21.9
0.5	29.7
0.6	38.35
0.7	48
0.8	58.45
0.9	69.7
1	82.1
1.1	95.5

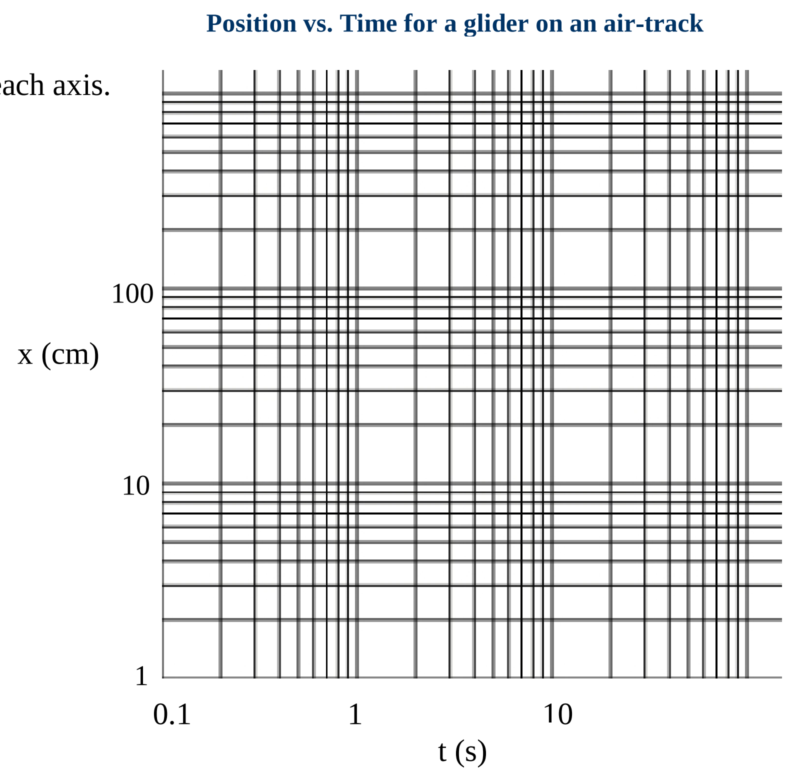


Labeling the Log axes:

The x-values range from 4.1cm to 95.5cm, so the vertical scale on the log paper needs to cover the numbers from 1 to 100.
Thus we can label the vertical scale using 10^0, 10^1, 10^2, or { **1** **10** **100** }.

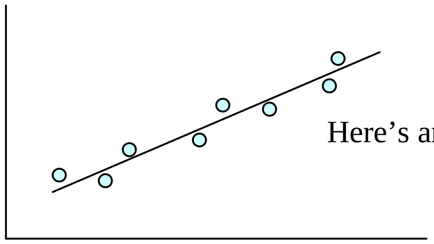
The t-values range from 0.1s to 1.1s, so the horizontal scale on the log paper needs to cover the numbers from 0.1 to *more than* 1.
Thus we can label the horizontal scale using 10^{-1}, 10^0, 10^1, or { **0.1** **1.0** **10** }

Remember to include the units for each axis.



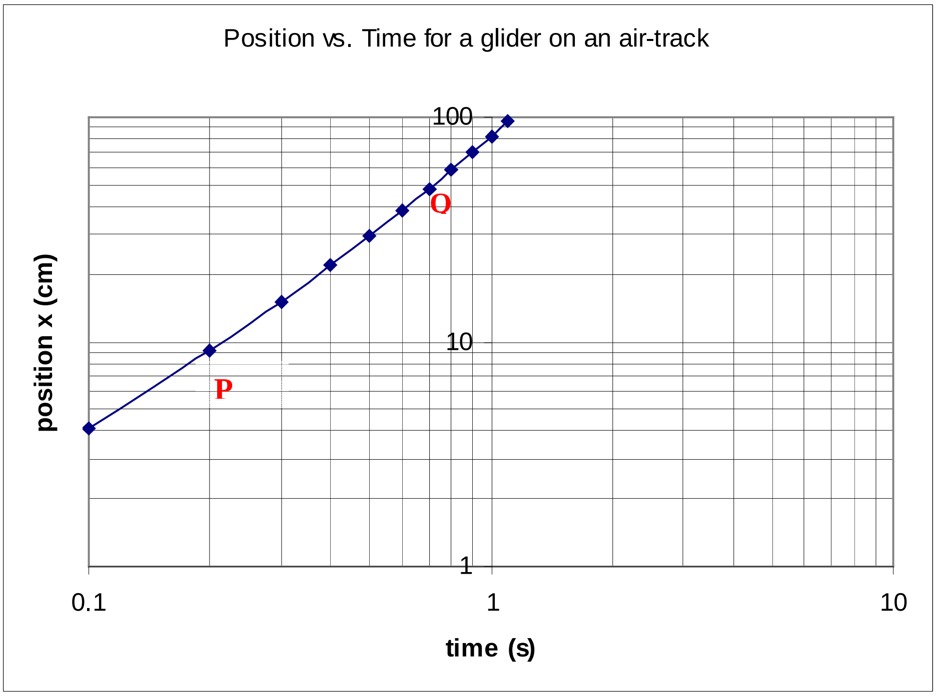
Step 2: Plotting the Data on the Log-Log Paper:

Remember that the log paper takes the mathematical log of the data for you. This means that you *do not* have to take the log of your data with your calculator, the paper does it for you!
Plot your data values on the log paper and draw a best fit straight line. (*Note: not all of your data points may touch your line. You may even find that NONE of your data points are on your line.)



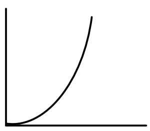
Here's an example of when *none* of the data points fall on the best fit line.

Time t (s)	Position x (cm)
0.1	4.1
0.2	9.1
0.3	15.1
0.4	21.9
0.5	29.7
0.6	38.35
0.7	48
0.8	58.45
0.9	69.7
1	82.1
1.1	95.5



Remember what the log paper has done for you; it took a function having the form

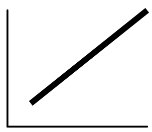
$x = kt^n$,



and stripped the exponent *n* off the ***t*** by taking the log of both sides to yield

$\log x \quad = \quad n \log t \quad + \quad \log k$ which has the familiar form

$y \quad = \quad m \quad x \quad + \quad b$ which is now a straight line on the log paper.



And what can you do with a straight line? You can easily find the slope 'm' and intercept 'b'!

Step 3: Finding the exponent '*n*' in $x = k\,t^{n}$ - It's just the slope of your line on the log paper.

Care must be taken when calculating the slope on log paper. Remember that you did not take the log of your data using your calculator, rather the logarithmic graph paper took the log for you! Thus, any number taken <u>off</u> the graph comes off the graph paper as the log of that number.

Method: Pick two points ***on your line*** : (x_2, t_2) and (x_1, t_1)**.** The previous example illustrates that your data points may not all be on the line. Even so, *you* must pick two points **on your line**!

$$\textit{the slope}: \quad n = \frac{\log(x_2) - \log(x_1)}{\log(t_2) - \log(t_1)} = \frac{\log\left(\frac{x_2}{x_1}\right)}{\log\left(\frac{t_2}{t_1}\right)}$$

For this example, I will use points Q (48, 0.7) and P (9.1, 0.2) to calculate the slope. (* These just happen to fall on *my* line. Remember that *your* data points may not all be on the line. Even so, *you* must pick two points **on your line**!)

$$\textit{the slope}: \quad n = \frac{\log(48) - \log(9.1)}{\log(0.7) - \log(0.2)} = \frac{\log\left(\frac{48}{9.1}\right)}{\log\left(\frac{0.7}{0.2}\right)} = 1.3$$

So your exponent *n* is 1.3.

So the data has the form $x = k\,t^{1.3}$. Now find the intercept 'k'.

Step 4: Determine the intercept *k* in $x = k\,t^{1.3}$.

Recall that you had $\log x = n \log t + \log k$, but since $n = 1.3$,

it now looks like $\boxed{\log x} = 1.3\ \boxed{\log t} + \boxed{\log k}$,

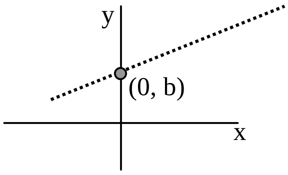
which has the familiar form: $y = m\ x + b$

Just do the same as you would when solving for *b* in y = mx + b, namely, set x = 0 to get y = b.

Setting x = 0 leads to

$y = 1.3x + b \quad \rightarrow \quad y = 1.3 * 0 + b$

which gives $y = b$.



But we have $\log x = 1.3 \log t + \log k$.

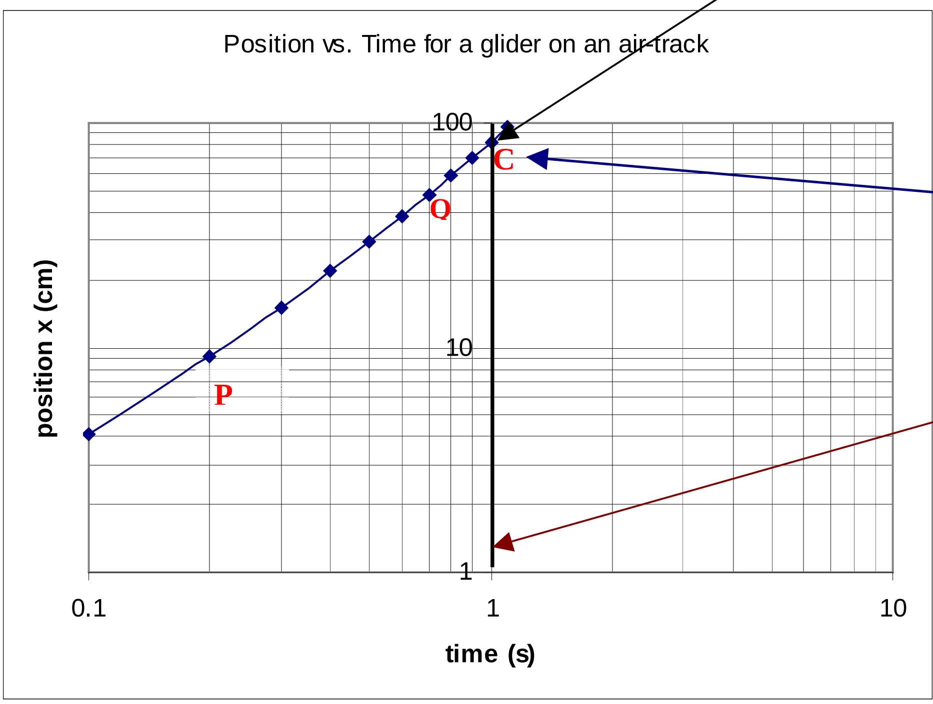
So setting $1.3 \log t = 0$ leaves

$\log x = 0 + \log k$ or

$\log x = \log k$

Recall $1.3 \log t = 0$ when $t = 1$!

Now look to see where your line intercepts the vertical x-axis when $t = 1$ (not when t = 0, since log 0 is undefined there!)



The point where your line intercepts the vertical $t = 1$ axis is your value of k!

This is the t = 1 axis.

In the graph above, the line intercepts the vertical x-axis at point **C**, where the vertical x-value is 82.1.

Thus the x-intercept is log 82.1.

Remember that it is *not* just 82.1 since the vertical axis is logarithmic, it's actually *log 82.1* when you take it off the paper.

so log x = log k becomes

log 82.1 = log k,

and thus $k = 82.1$. It's that easy. You didn't even need a calculator to find k.

Step 5: Putting it all together.

We were trying to fit the data to the form $x = k\,t^n$.

We have found that n = 1.3 *and* k = 82.1.

Therefore, we can write that this data fits the equation

$x = 82.1 t^{1.3}$. But wait!

You must include the proper units with the value of k. To find them, simply rearrange the equation $x = k\,t^n$ to solve for k, in other words,

$$k = \frac{x}{t^n}$$

Since x has units of centimeters and t has units of seconds, the units for k must be $\frac{cm}{s^n}$.

In this exercise where n = 1.3, k has units of $\frac{cm}{s^{1.3}}$. So to fully express the function, we have:

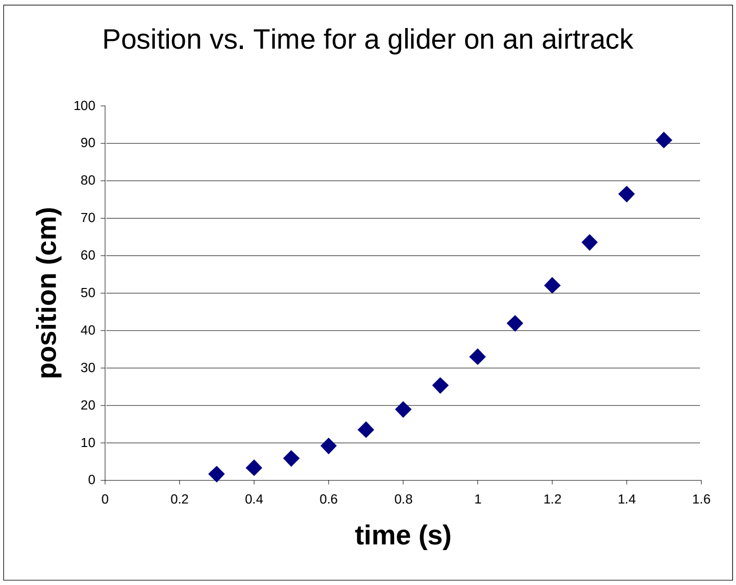
$$x = 82.1 \frac{cm}{s^{1.3}} t^{1.3}.$$

Graphing on log paper is a powerful tool to have in your engineering toolbox. To refine your skills, we will be doing several of these graphs throughout the semester. All of them will be used to find the constant k and exponent n for data that we suspect is non-linear and follows a $x = k\,t^n$ trend.
Now you will get another chance to practice the skill of graphing on log paper.

Name:

Here is a student's data for a trial run of a glider moving down a tilted airtrack.

Time t (s)	Position x(cm)
0.3	1.626
0.4	3.339
0.5	5.833
0.6	9.202
0.7	13.528
0.8	18.890
0.9	25.358
1	33.000
1.1	41.878
1.2	52.0555
1.3	63.587
1.4	76.530
1.5	90.937



It appears that the data obeys a power law of the form $x = k t^n$, where *n* and *k* are constants to be determined. You will use the techniques outlined in this tutorial to determine the constants *k* and *n*. Please show all of your work on a separate piece of paper.

On the 2-cycle log-log paper provided, construct a complete and detailed graph of position verses time for the glider. Please make sure your graph is titled and labeled correctly.

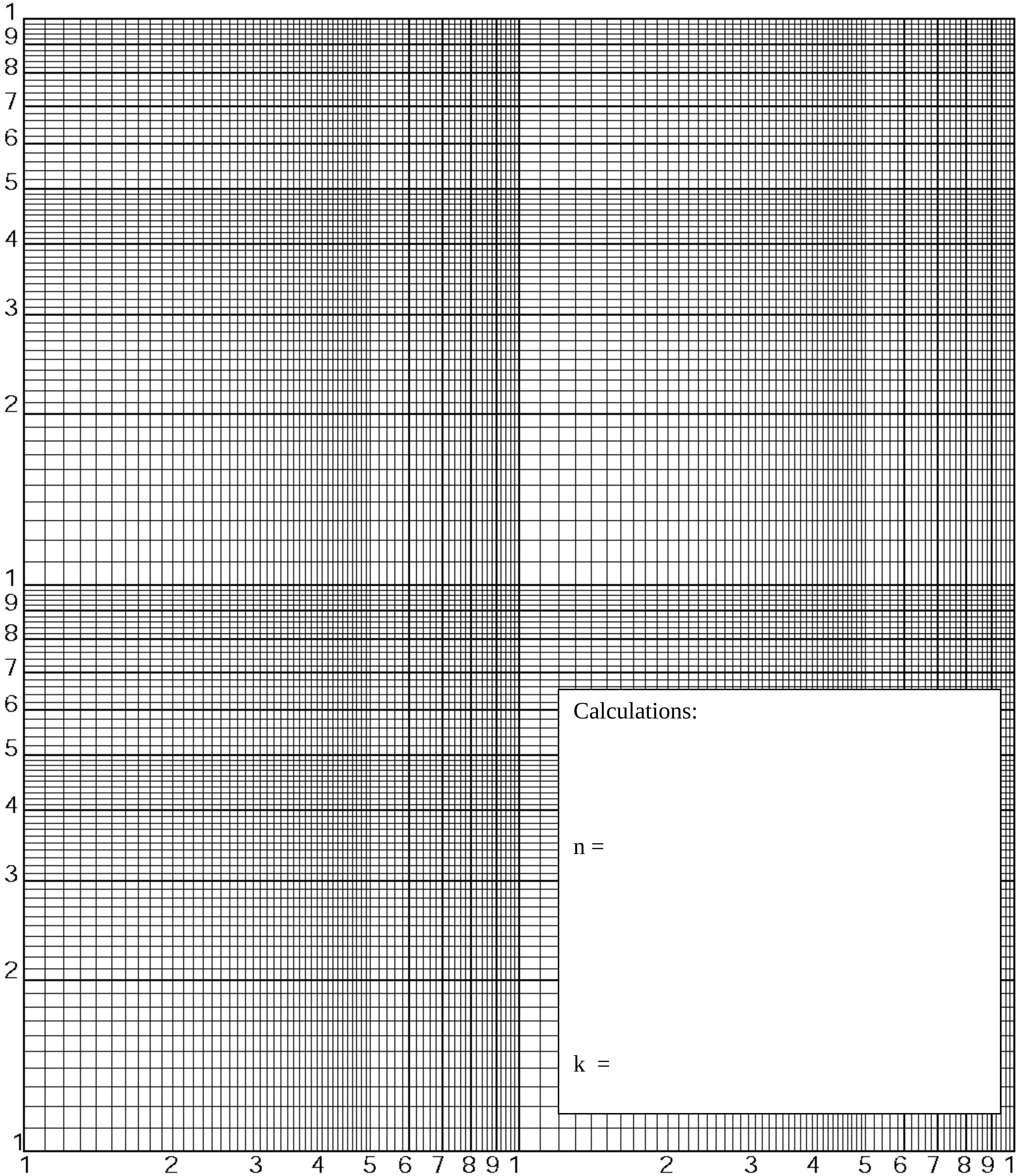
Please use the step-by-step tutorial in the previous sections to assist in the construction of the graph. If you ask me "how do I make a log-log graph?", I will refer you to this entire tutorial.

After drawing a best-fit line on your graph, determine the value of the exponent *n* by finding the slope of your line. Please show all of your work in the box provided on the graph paper.
Now use your graph to determine the constant *k*. Again, use the step-by-step tutorial to do this. If you ask me "how do I find *n* and *k*?", I will refer you to this entire tutorial.

Using appropriate units, express the position of the glider as a function of time using your values of *k* and *n*.

x (t) = ________________

Please hand this page in to me when I ask for it during class.



Calculations:

n =

k =